Modal Identification of Gyroscopic Distributed-Parameter Systems

Mark A. Norris*
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
and
Larry Silverberg†
North Carolina State University, Raleigh, North Carolina 27695

A new modal identification method for gyroscopic distributed-parameter systems is presented. The method represents an extension of previous work for the class of self-adjoint distributed-parameter systems. The modal identification method is formulated as a variational problem in which stationary values of a functional quotient are sought. The computation of the functional quotient is carried out using a set of admissible functions defined over the spatial domain of the system. As an illustration, the modal identification of a whirling shaft undergoing bending vibration is carried out and the effectiveness of the method is verified.

Introduction

STRUCTURAL dynamicists, especially those involved in vibration testing, are concerned with identifying the natural frequencies of oscillation and the associated natural modes of vibration of a structure. This paper describes a new method of modal identification which applies to gyroscopic distributed-parameter systems. These systems arise in structures which either rotate or possess components that rotate such as helicopter blades, rotors, whirling shafts, and spinning satellites. Gyroscopic distributed-parameter systems (DPS) are conveniently described in terms of rotating sets of axes and the vibration represents perturbations in the motion relative to these sets of axes. The modal identification method introduced in this paper is an extension of previous work which applied to nongyroscopic systems.[1-3] The modal identification method proposed in this paper is formulated as a variational problem in which stationary values of a functional quotient are sought. The parameters contained in the functional quotient are obtained from measurements of the transient response taken at multiple points over the structure.

Most modal identification techniques are applicable to non-self-adjoint systems, which does not preclude their applicability to gyroscopic systems. Examples of these are the Ibrahim time-domain (ITD) method,[4,5] the polyreference technique,[6] and the eigensystem realization algorithm (ERA).[7] Non-self-adjointness properties in structures are caused by damping and/or circulatory forces, which implies that the motion of the structure cannot be expressed as a linear combination of normal modes, as the eigensolution is complex. The methods for identification of non-self-adjoint systems may be applied to nearly all linear systems, to include identifying complex eigenvalues and eigenvectors. Gyroscopic systems represent a class of linear non-self-adjoint systems in which the damping operator is skew-symmetric. This implies that the eigenvalues of gyroscopic systems consist of pairs of pure imaginary complex conjugates, and correspondingly, the eigenvectors occur in pairs of complex conjugates whose real and imaginary parts are mutually orthogonal.[8]

The modal identification method proposed in this paper is formulated directly for gyroscopic distributed-parameter systems. The identification algorithm takes advantage of the gyroscopic behavior of the system and so the identification becomes less sensitive to sensor noise and unknown external disturbances. In addition, the method is formulated directly for distributed-parameter systems which implies that an infinite number of natural frequencies and associated natural modes can be identified. In practice, however, the use of discrete (in-space) measurements prevents us from identifying all of the modal quantities.

The modal identification techniques introduced here follow a typical realization process in which the modes significantly contributing to the system response are targeted for identification. For example, the ERA uses the singular-value decomposition in order to obtain the minimum-order system realization, and then performs the modal identification. Although system realization will not be discussed further in this paper, note that the variational modal identification method, from which the technique introduced here is based, incorporates system realization and represents a constrained ERA for lightly damped nearly self-adjoint systems.[9]

The natural frequencies and associated natural modes of vibration in gyroscopic distributed-parameter systems can be computed from a distributed-parameter eigenvalue problem. Equivalently, a variational formulation can be considered in which the natural frequencies and natural modes of vibration are computed from a functional quotient known as Rayleigh's quotient.[8] The stationary values of Rayleigh's quotient are related to the natural frequencies and occur when Rayleigh's quotient is evaluated at the natural modes of vibration. The modal identification method described in this paper is formulated as a variational problem in which another functional quotient is defined in a form that is particularly suited for modal identification. The exact evaluation of this functional quotient requires distributed measurements (in-space). However, distributed measurements are often unavailable so that one must resort to using discrete measurements instead. This presents no difficulty provided the number of discrete measurements is not less than the number of modes significantly participating in the overall system response.[3]

Equation of Motion for Gyroscopic Distributed-Parameter Systems

The motion of gyroscopic distributed-parameter systems (DPS) can be described by the partial differential equation

Received Feb. 24, 1989; revision received Jan. 16, 1990.

*Assistant Professor, Engineering Science and Mechanics; currently Senior Engineering Specialist, Thomas Lord Research Center. Member AIAA.

†Associate Professor, Mars Mission Research Center. Member AIAA.

(PDE)[10]:

$$ku + g\dot{u} + m\ddot{u} = f \tag{1}$$

with

- $u = u(P,t)$, displacement vector of spatial point P at time t
- $k = k(P)$, positive definite linear differential matrix stiffness operator of order $2p$
- $g = g(P)$, skew-symmetric linear differential matrix gyroscopic operator
- $m = m(P)$, mass distribution
- $f = f(P,t)$, external force distribution
- $(\dot{\ })$ = differentiation with respect to time

The displacement u and the external force distribution f are functions in the configuration space V satisfying the homogeneous geometric boundary conditions. The geometric boundary conditions are of orders no greater then p. The inner product between functions in V is given by[10]

$$[\xi_1,\xi_2] = \int_D \xi_1^T \xi_2 \, dD$$

where D is the spatial domain of the system and $\xi_1 = \xi_1(P,t)$ and $\xi_2 = \xi_2(P,t)$ are any two functions in V.

It will prove advantageous to rewrite the equations of motion, Eq. (1), in the state space $W = V \times V$ as

$$I\dot{w} + Gw = F \tag{2}$$

with w, F, I, and G defined as follows:

State vector of spatial point P at time t

$$w = w(P,t) = [\dot{u}^T(P,t) \; u^T(P,t)]^T$$

External force distribution:

$$F = F(P,t) = [f^T(P,t) \; 0^T]^T$$

Self-adjoint positive definite matrix differential operator:

$$I = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$$

Skew symmetric matrix differential operator:

$$G = \begin{bmatrix} g & +k \\ -k & 0 \end{bmatrix}$$

The inner product between functions in W is given by

$$(\eta_1,\eta_2) = [\eta_{11},\eta_{21}] + [\eta_{12},\eta_{22}]$$

where

$$\eta_1 = \eta_1(P,t) = [\eta_{11}^T(P,t) \; \eta_{12}^T P,t)]^T$$

$$\eta_2 = \eta_2(P,t) = [\eta_{21}^T(P,t) \; \eta_{22}^T(P,t)]^T$$

are any two functions in W.

The eigenvalue problem associated with Eq. (2) is given by

$$i\omega_r I\phi_r + G\phi_r = 0 \tag{3}$$

where $\phi_r = \phi_r(P)$ is the rth natural mode of vibration and ω_r is the rth natural frequency.

The natural modes of vibration are complex and the associated natural frequencies are real and positive. The real and imaginary parts of the natural modes of vibration are mutually orthogonal and satisfy the spatial orthonormality conditions[8]:

$$(\sigma_r, I\sigma_s) = \delta_{rs} \qquad (\tau_r, I\tau_s) = \delta_{rs}$$
$$(\tau_r, G\sigma_s) = \omega_r\delta_{rs} \qquad (\sigma_r, I\tau_s) = 0 \tag{4}$$

where

$$\sigma_r = \sigma_r(P), \qquad \tau_r = \tau_r(P)$$

are the real and imaginary parts of $\phi_r(P)$, respectively.

In addition, the real and imaginary parts of the natural modes of vibration satisfy the eigenvalue problem[8]:

$$\omega_r^2 I\sigma_r = L\sigma_r, \qquad \omega_r^2 I\tau_r = L\tau_r \tag{5}$$

where $L = G^T I^{-1} G$ is the positive definite, self-adjoint matrix differential operator.

From Eq. (5), both the real and imaginary parts of the natural modes of vibration represent solutions to the same eigenvalue problem. Therefore, the solution of the eigenvalue problem is comprised of an infinite number of twice-repeated natural frequencies and the associated real and imaginary parts of the natural modes of vibration. The real and imaginary parts of the rth natural mode are linearly independent and can be orthogonalized. Then from Eq. (5), we obtain additional spatial orthogonality conditions in the form:

$$(\sigma_r, L\sigma_s) = \omega_r^2\delta_{rs}, \quad (\tau_r, L\tau_s) = \omega_r^2\delta_{rs}, \quad (\sigma_r, L\tau_s) = 0 \tag{6}$$

The natural modes of vibration span the vector space W. Therefore, both the state vector and the external force distribution can be expressed in terms of the natural modes as follows:

$$w = \sum_{r=1}^{\infty}(\sigma_r a_r + \tau_r b_r), \quad a_r = (\sigma_r, Iw), \quad b_r = (\tau_r, Iw) \tag{7a}$$

$$F = \sum_{r=1}^{\infty} I(\sigma_r A_r + \tau_r B_r), \quad A_r = (\sigma_r, F), \quad B_r = (\tau_r, F) \tag{7b}$$

with $a_r = a_r(t)$, $b_r = b_r(t)$ being the real and imaginary parts of the rth modal displacement, respectively, and $A_r = A_r(t)$, $B_r = B_r(t)$ the real and imaginary parts of the rth modal force, respectively.

Introducing Eq. (7a) into Eq. (2) and considering Eq. (4) yields the real and imaginary parts of the modal equations of motion:

$$\dot{a}_r + \omega_r b_r = A_r, \qquad \dot{b}_r + \omega_r a_r = B_r \tag{8}$$

where $r = 1,2,\ldots$.

Equation (8) represents an infinite set of pairs of independent first-order ordinary differential equations. Rather than working with the equations of motion, Eq. (1), in the configuration space V or in the state space W, we have now transformed our equations into the modal space X. Indeed, Eq. (8) represents the linear transformation of Eq. (2) in which functions w and F have been transformed into the sequence of functions $[a_1, a_2, \ldots, b_1, b_2, \ldots]$ and $[A_1, A_2, \ldots, B_1, B_2, \ldots]$, respectively. The inner product between functions in X is given by

$$<\theta_1,\theta_2> = \frac{1}{T}\int_0^T \theta_1\theta_2 \, dt \tag{9}$$

where $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$ are any two functions in X.

In the limit as T approaches infinity and in the absence of an external force distribution, or equivalently, in the absence of modal forces, the modal displacements satisfy the temporal orthogonality conditions:

$$\langle a_r, a_s \rangle = M_r \delta_{rs}, \qquad \langle b_r, b_s \rangle = N_r \delta_{rs}, \qquad \langle a_r, b_s \rangle = 0 \tag{10a}$$

$$\langle \dot{a}_r, \dot{a}_s \rangle = M_r \omega_r^2 \delta_{rs}, \qquad \langle \dot{b}_r, \dot{b}_s \rangle = N_r \omega_r^2 \delta_{rs}, \qquad \langle \dot{a}_r, \dot{b}_s \rangle = 0 \tag{10b}$$

where M_r and N_r are constants depending on the initial conditions.

Variational Formulation for the Eigenvalue Problem

A variational formulation for the eigenvalue problem provided the motivation for the variational modal identification method introduced in the next section of this paper. Therefore, in this section we first review a variational formulation for the eigenvalue problem. The variational formulation consists of replacing the solution of the eigenvalue problems, Eq. (5), with seeking stationary values of the functional quotient, known as Rayleigh's quotient, given by

$$R = \frac{(L^{1/2}\boldsymbol{\phi}, L^{1/2}\boldsymbol{\phi})}{(\boldsymbol{\phi}, I\boldsymbol{\phi})} \tag{11}$$

where $\boldsymbol{\phi} = \boldsymbol{\phi}(P)$ is any function in W and $R = R(\boldsymbol{\phi})$ is Rayleigh's quotient.

The symmetric form of the numerator can be obtained through an integration by parts of $(\boldsymbol{\phi}, L\boldsymbol{\phi})$ while considering the full set of boundary conditions (geometric and natural boundary conditions).[10]

The immediate interest lies in showing that seeking stationary values of R is equivalent to solving Eq. (5). To that end, $\boldsymbol{\phi}$ can be expressed in terms of the natural modes of vibration as

$$\boldsymbol{\phi} = \sum_{r=1}^{\infty} (\boldsymbol{\sigma}_r \alpha_r + \boldsymbol{\tau}_r \beta_r) \tag{12}$$

where α_r and β_r are rth undetermined coefficients.

Introducing Eq. (12) into Eq. (11) while considering Eqs. (4) and (6) yields

$$R = \frac{\sum_{r=1}^{\infty} (\alpha_r^2 + \beta_r^2)\omega_r^2}{\sum_{r=1}^{\infty} (\alpha_r^2 + \beta_r^2)} \tag{13}$$

where $R = R(\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots)$ is the functional quotient as a function of the undetermined coefficients.

At stationary values of R, the first variation of R vanishes, so

$$\mathrm{d}R = \sum_{r=1}^{\infty} \left(\frac{\partial R}{\partial \alpha_r} \mathrm{d}\alpha_r + \frac{\partial R}{\partial \beta_r} \mathrm{d}\beta_r \right) = 0 \tag{14}$$

implying that

$$\frac{\partial R}{\partial \alpha_r} = \frac{2\alpha_r(\omega_r^2 - R)}{\sum_{s=1}^{\infty} (\alpha_s^2 + \beta_s^2)} = 0 \tag{15a}$$

$$\frac{\partial R}{\partial \beta_r} = \frac{2\beta_r(\omega_r^2 - R)}{\sum_{s=1}^{\infty} (\alpha_s^2 + \beta_s^2)} = 0 \tag{15b}$$

From Eq. (15), we obtain the solutions $R^{(s)} = \omega_s^2$ and $\alpha_r^{(s)} = \beta_r^{(s)} = \delta_{rs}$ $(r,s = 1, 2, \ldots)$. Indeed, the sth stationary value of the functional quotient $R^{(s)}$ is identical to the sth natural frequency squared and from Eq. (12) the stationarity occurs when the function $\boldsymbol{\phi}^{(s)}$ is a linear combination of the real and imaginary parts of the sth natural mode of vibration, $\boldsymbol{\sigma}_s$ and $\boldsymbol{\tau}_s$.

Variational Formulation for the Modal Identification of Gyroscopic Distributed-Parameter Systems

Our interest lies in identifying the natural frequencies and associated natural modes of vibration *not* via the variational formulation of the previous section, but rather from the transient response. To that end, the following functional quotient is defined in a form particularly suitable for modal identification:

$$S = \frac{\langle \dot{y}, \dot{y} \rangle}{\langle y, y \rangle}, \qquad y = (I\boldsymbol{\Psi}, \boldsymbol{w}) \tag{16}$$

where $S = S(\boldsymbol{\Psi})$ is the functional quotient suitable for modal identification, $y = y(t)$ is the generalized state, and $\boldsymbol{\Psi} = \boldsymbol{\Psi}(P)$ is any admissible function in the state space W.

First, we show that stationary values of S yield the natural frequencies and associated natural modes of vibration. To that end, $\boldsymbol{\Psi}$ in Eq. (16) is expressed in terms of the real and imaginary parts of the natural modes as

$$\boldsymbol{\Psi} = \sum_{r=1}^{\infty} (\boldsymbol{\sigma}_r \gamma_r + \boldsymbol{\tau}_r \delta_r) \tag{17}$$

where γ_r and δ_r are the rth undetermined coefficients.

Introducing Eq. (17) into Eq. (16) and using Eqs. (4), (7a), and (10) yields

$$S = \frac{\sum_{r=1}^{\infty} \omega_r^2 (\gamma_r^2 M_r + \delta_r^2 N_r)}{\sum_{r=1}^{\infty} (\gamma_r^2 M_r + \delta_r^2 N_r)} \tag{18}$$

where $S = S(\gamma_1, \gamma_2, \ldots, \delta_1, \delta_2, \ldots)$ is the functional quotient as a function of the undetermined coefficients.

At stationary values of S, the first variation of S vanishes so that

$$\mathrm{d}S = \sum_{r=1}^{\infty} \left(\frac{\partial R}{\partial \gamma_r} \mathrm{d}\gamma_r + \frac{\partial R}{\partial \delta_r} \mathrm{d}\delta_r \right) = 0$$

implying that

$$\frac{\partial S}{\partial \gamma_r} = \frac{2\gamma_r M_r(\omega_r^2 - S)}{\sum_{s=1}^{\infty} (\gamma_s^2 M_s + \delta_s^2 N_s)} = 0 \tag{19a}$$

$$\frac{\partial S}{\partial \delta_r} = \frac{2\delta_r N_r(\omega_r^2 - S)}{\sum_{s=1}^{\infty} (\gamma_s^2 M_s + \delta_s^2 N_s)} = 0 \tag{19b}$$

From Eq. (19), we obtain the solutions $S^{(s)} = \omega_s^2$ and $\gamma_r^{(s)} = \delta_r^{(s)} = \delta_{rs}$ $(s = 1, 2, \ldots)$. Indeed, the sth stationary value of the functional quotient $S^{(s)}$ is identical to the sth natural frequency squared and from Eq. (17) the stationarity occurs when the function $\boldsymbol{\Psi}^{(s)}$ is a linear combination of the real and imaginary parts of the sth natural mode of vibration, $\boldsymbol{\sigma}_s$ and $\boldsymbol{\tau}_s$.

Discretization

The exact evaluation of the functional quotient, Eq. (16), implies that distributed measurements are required. Indeed, the generalized state y is obtained from the distributed state $\boldsymbol{w}$. However, in practice, distributed measurements are often not feasible so that one resorts to using discrete measurements. The implication is that the stationary values of S in Eq. (16) cannot be evaluated exactly. Formally, the replacement of distributed measurements with discrete measurements is a discretization process. The following describes the discretization process in more detail. Referring to Eq. (16), the functions $\boldsymbol{\Psi}$ and $\boldsymbol{w}$ are constrained to within the subspace W_n of W, generated by the given admissible functions $\boldsymbol{\Psi}_r (r = 1, 2, \ldots, n)$ in

W, so that

$$\Psi = \sum_{r=1}^{n} \Psi_r c_r \tag{20a}$$

$$w = \sum_{r=1}^{n} \Psi_r d_r \tag{20b}$$

where c_r is the rth undetermined coefficient, $d_r = d_r(t)$ is the rth undetermined generalized coefficient, and $\Psi_r = \Psi_r(P)$ is the rth admissible function in W.

Introducing Eq. (20) into Eq. (16), we obtain

$$S = \frac{\sum_{r=1}^{n}\sum_{s=1}^{n} k_{rs} c_r c_s}{\sum_{r=1}^{n}\sum_{s=1}^{n} m_{rs} c_r c_s} \tag{21a}$$

$$y = \sum_{r=1}^{n}\sum_{s=1}^{n} \psi_{rs} c_r d_s \tag{21b}$$

where

$$\psi_{rs} = (I\psi_r, \psi_s) \tag{22a}$$

$$k_{rs} = \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ri}\psi_{sj}\langle \dot{d}_i, \dot{d}_j\rangle \tag{22b}$$

$$m_{rs} = \sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ri}\psi_{sj}\langle d_i, d_j\rangle \tag{22c}$$

Seeking stationary values of the functional quotient, Eq. (21a), is equivalent to solving the nth-order eigenvalue problem:

$$\omega_r^2 M c^{(r)} = K c^{(r)} \tag{23}$$

where $M = (m_{rs})$ is the $n \times n$ identified mass matrix, $K = (k_{rs})$ is the $n \times n$ identified stiffness matrix, and $c^{(r)} = [c_1^r\ c_2^r \ldots]^T$ is the rth n-dimensional vector of undetermined coefficients.

Then, from Eq. (23), we obtain the linear combination of the real and imaginary parts of the rth natural mode of vibration:

$$\Psi^{(r)} = \epsilon_{1r}\sigma_r + \epsilon_{2r}\tau_r = \sum_{s=1}^{n} \Psi_s c_s^{(r)} \tag{24}$$

where ϵ_{1r} and ϵ_{2r} are arbitrary constants.

The accuracy of the approximation depends on the choice of admissible functions. The next section describes the discretization process in terms of a specific example.

Variational Modal Identification of a Whirling Shaft

As an illustration of variational modal identification of gyroscopic distributed-parameter systems, we consider identifying the natural frequencies of oscillation of the whirling shaft undergoing bending vibration as shown in Fig. 1. The numerical simulation assumes that the shaft has rigid pin supports, contains no internal or external damping, and has no rotating unbalances or eccentricities. Furthermore, the static deflection due to gravity is neglected. The system is symmetric; hence, the whirling and rotational speeds are identical.[11] The shaft motion is governed by PDE in the form of Eq. (1) described by[10]

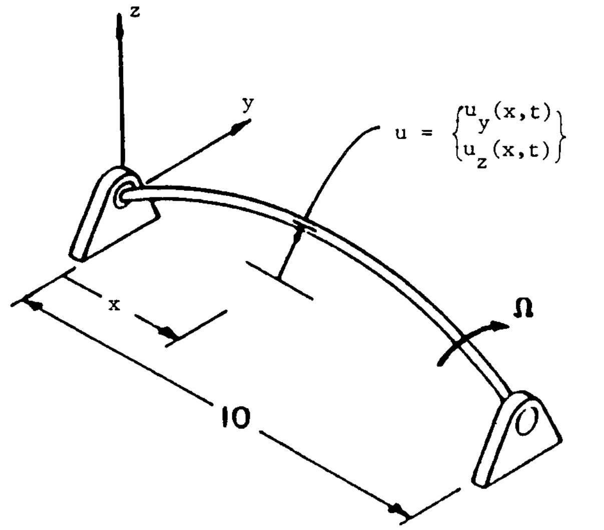

Fig. 1 Uniform shaft whirling at angular velocity Ω.

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2}{\partial x^2}\right] - m\Omega^2 & 0 \\ 0 & \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2}{\partial x^2}\right] - m\Omega^2 \end{bmatrix}\begin{Bmatrix} u_y \\ u_z \end{Bmatrix} + \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix}\begin{Bmatrix} \dot{u}_y \\ \dot{u}_z \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_y \\ \ddot{u}_z \end{Bmatrix} = 0 \tag{25}$$

where $u_y = u_y(x,t)$ and $u_z = u_z(x,t)$ represent displacements in the y and z directions, respectively, referred to a body-fixed reference frame, $m = m(x)$ is the mass distribution, $EI = EI(x)$ is the bending rigidity, and Ω is the constant angular velocity of the shaft. We consider the case where the shaft is uniform so that $m = m_o$ and $EI = EI_o$, where m_o and EI_o are constants. The differential operators I and G in Eq. (2) are then

$$I = \begin{bmatrix} m_o & 0 & 0 & 0 \\ 0 & m_o & 0 & 0 \\ 0 & 0 & EI_o\frac{\partial^4}{\partial x^4} - m_o\Omega^2 & 0 \\ 0 & 0 & 0 & EI_o\frac{\partial^4}{\partial x^4} - m_o\Omega^2 \end{bmatrix} \tag{26a}$$

$$G = \begin{bmatrix} 0 & -2m_o\Omega & EI_o\frac{\partial^4}{\partial x^4} - m_o\Omega^2 & 0 \\ 2m_o\Omega & 0 & 0 & EI_o\frac{\partial^4}{\partial x^4} - m_o\Omega^2 \\ m_o\Omega^2 - EI_o\frac{\partial^4}{\partial x^4} & 0 & 0 & 0 \\ 0 & m_o\Omega^2 - EI_o\frac{\partial^4}{\partial x^4} & 0 & 0 \end{bmatrix} \tag{26b}$$

For convenience, we choose $m_o = 1$ kg/m, $EI_o = 1$ N · m², $\Omega = 1$ rad/s, and the shaft length $L = 1$ m. We discretize the system and use as admissible functions $\Psi_r (r = 1,2, \ldots, 20)$ in Eq. (20)[10]:

$$\Psi_r = [1\ 0\ 0\ 0]^T \phi_r, \qquad r = 1, 2, \ldots, 5 \tag{27a}$$

$$\Psi_r = [0\ 1\ 0\ 0]^T \phi_{r-5}, \qquad r = 6, 7, \ldots, 10 \tag{27b}$$

$$\Psi_r = [0\ 0\ 1\ 0]^T \phi_{r-10}, \qquad r = 11, 12, \ldots, 15 \tag{27c}$$

$$\Psi_r = [0\ 0\ 0\ 1]^T \phi_{r-15}, \qquad r = 16, 17, \ldots, 20 \tag{27d}$$

where $n = 20$ and $\phi_r = \sqrt{2} \sin(r\pi x)$ $(r = 1, 2, \ldots, 5)$ are the orthonormal set of modes for the nonrotating shaft, with corresponding natural frequencies ω_{or}, in which $[\phi_r, \phi_s] = \delta_{rs}$,

$$\left[\phi_r, \frac{d^4\phi_s}{dx^4}\right] = \omega_{or}^2 \delta_{rs}, \text{ and } \omega_{or} = \left(\frac{r\pi}{L}\right)^2$$

The four sets of admissible functions Ψ_r in Eq. (27) correspond to displacement in the y direction, displacement in the z direction, velocity in the y direction, and velocity in the z direction. The motion of the discretized rotating shaft in both the y and z directions is spanned by the first five eigenfunctions of the nonrotating shaft. Note that the functions Ψ_r satisfy all of the boundary conditions of the whirling shaft, although it is only necessary to satisfy the geometric boundary conditions (i.e., zero displacement at $x = 0,L$) in order to qualify as admissible functions and even this restriction can be relaxed for identification purposes.[2] Using the admissible functions in Eq. (27), the solution of the algebraic eigenvalue problems by Eq. (5) provides the first ten pairs of natural frequencies of the rotating shaft. The frequencies are displayed in Table 1. Note that the pairs of natural frequencies are exact and are given by $\omega_{ol} - \Omega$ and $\omega_{ol} + \Omega$ $(l = 1,2, \ldots, 5)$, corresponding to asynchronous and synchronous precession, respectively.[12]

We wish to identify the first ten pairs of natural frequencies using five discrete sensors measuring displacements u_y and u_z, velocities $\dot{u}_y$ and $\dot{u}_z$, and accelerations $\ddot{u}_y$ and $\ddot{u}_z$, relative to the body-fixed reference frame at locations spaced equally along the shaft. This can be accomplished using strain gages from which displacements relative to the body-fixed coordinate system are synthesized by a differentiation with respect to time to provide velocities and then a second differentiation to provide

Table 1 Actual and identified natural frequencies of the whirling shaft

Actual natural frequencies, (rad/s)	Identified natural frequencies, $T = 5$ s, (rad/s)	Identified natural frequencies, $T = 30$ s, (rad/s)
8.870	8.797302	8.841059
8.870	8.877332	8.896691
10.870	10.707875	10.851407
10.870	11.106201	10.889549
38.478	38.332860	38.444371
38.478	38.586107	38.511607
40.478	40.284692	40.445049
40.478	40.730495	40.513077
87.826	87.599280	87.792785
87.826	87.971269	87.858535
89.826	89.852979	89.803854
89.826	89.897615	89.850953
156.914	156.573654	156.862875
156.914	157.137286	156.963512
158.914	158.964261	158.861168
158.914	158.995041	158.967460
245.740	245.533687	245.618394
245.740	245.533941	245.855393
247.740	247.318715	247.650114
247.740	248.711974	247.837482

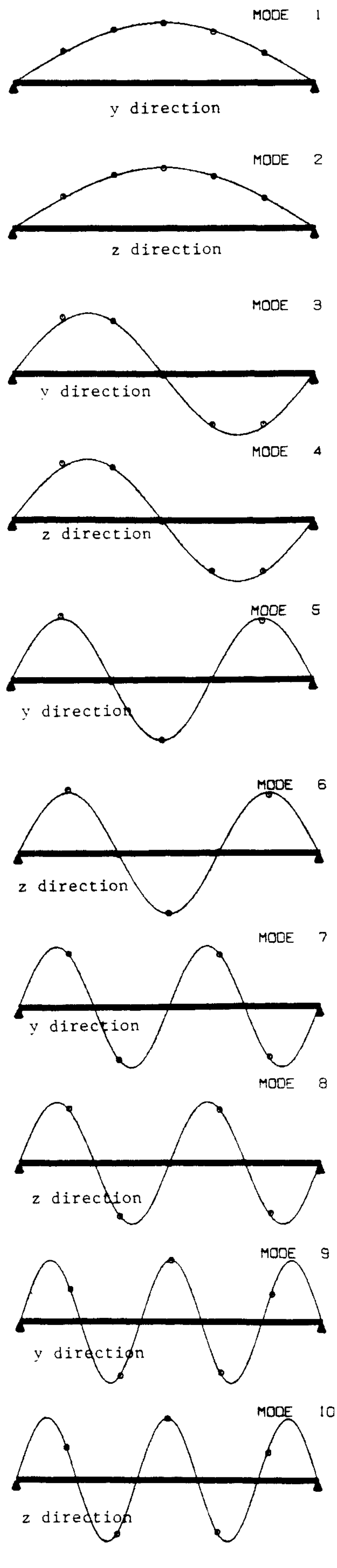

Fig. 2 The magnitudes of the actual and identified complex mode shapes.

accelerations. In practice, differentiation with respect to time is highly sensitive to sensor noise, thus requiring a form of temporal filtering. When acceleration measurements relative to the body-fixed frame are available, temporal intergrations are carried out to obtain velocities and then displacements as in Ref. 3. Note that the indicated integrations must accompany the bias. In this illustration, it is assumed that a realization process has been conducted as in Refs. 7 and 9 and that the realization targets 20 modes for identification. The identification process uses the transient response of the shaft initially excited by an impulse of magnitude 1 N at $x = 10/11$ m in each of the y and z directions, which implies that the numerical simulation is completed with no external excitation with the exception of the rotor which imparts constant angular velocity to the shaft. Note that at this location, the first 20 modes will be excited, where the contribution of the higher modes to the response is ignored. Examining the contribution to the response of a single mode, every point on the shaft traces a circle in the y-z plane.[11] The transient response of the rotating shaft can be viewed as a linear combination of the modes of revolution multiplied by time-dependent coordinates.

The identification procedure consists of temporally correlating the transient response measurements using the collocation approximation in Eq. (22). The collocation approximation uses spatial Dirac delta functions in Eq. (16). These functions are defined to be zero over the domain of the structure excluding the sensor locations.[2] The identified mass and stiffness matrices in Eq. (23) are computed using

$$M = \langle y, y^T \rangle \tag{28a}$$

$$K = \langle \dot{y}, \dot{y} \rangle \tag{28b}$$

with

$$y = [u_y(x,t) \ldots u_y(x_5,t)\ u_z(x_1,t) \ldots u_z(x_5,t)\dot{u}_y(x_1,t) \ldots \dot{u}_y \times (x_5,t)\dot{u}_z(x_1,t) \ldots \dot{u}_z(x_5,t)]^T$$

The temporal inner products $<,>$ in Eq. (28) are computed using a finite time interval T, which implies that the temporal orthogonality conditions in Eq. (10) are approximated by the truncation of the infinite time interval. In order to illustrate the effects of truncation error, we evaluate the inner products using $T = 5$ s and $T = 30$ s with a sampling frequency of 100 Hz. The actual and identified natural frequencies are presented in Table 1. Because of truncation error, the identified natural frequencies do not occur precisely in pairs. Note that the average values can provide improved estimates. Furthermore, note that as the sampled time T increases, the results of the modal identification technique improve. The magnitudes of the actual and identified (using $T = 5$ s) complex mode shapes are shown in Fig. 2.

Conclusions

A new modal identification method for gyroscopic DPS is proposed. The method is formulated as a variational problem in which stationary values of a functional quotient are sought in the same manner in which the stationary values of Rayleigh's quotient are sought to determine the natural frequencies for self-adjoint DPS. The computation of the functional quotient is carried out using a set of admissible functions defined over the spatial domain of the system. The method is formulated specifically for DPS, whereas other modal identification algorithms are formulated specifically for finite-dimensional truncated systems. The natural frequencies of a whirling shaft undergoing bending vibration illustrates the method. Future work will verify experimentally the modal identification method introduced in this paper for gyroscopic distributed-parameter systems as a natural extension to the experimental verifications reported in Ref. 3 for self-adjoint distributed-parameter systems.

Acknowledgments

This research was supported in part by the ASEE Program, Summer Faculty Research, 1987, and monitored by R. Tolson of the Interdisciplinary Research Office, NASA Langley Research Center. Support was also provided by Grant NAGW-1331 of the Mars Mission Research Center. The authors sincerely appreciate Larry D. Mitchell of the Department of Mechanical Engineering, Virginia Polytechnic Institute and State University, for his help with the numerical example.

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